Lecture 16 (2/14/22).

- Recall and finish proof of Riemann Mapping Thm from Lecture 15 notes.

Factorization theorems.

Recall that if $f \in H(G)$ and has zeros at $a_1, \dots, a_n$, then $\exists g \in H(G)$ s.t

$$f(z) = (z-a_1)\cdots(z-a_n)\,g(z). \quad (*)$$

Moreover, any additional zeros will also be zeros of g. Thus, if the zero set of f is <u>finite</u>, $a_1, \dots, a_n$, repeated w/ multiplicities then $(*)$ holds and g has no zeros in G.

We also know that f may have ∞ many zeros (but without limit points in G).
Is there an analogue of $(*)$?

$$f(z) = \prod_{j=1}^{\infty} (z-a_j)\, g(z), \quad g \neq 0 \ ?$$

What do we mean by infinite product?

Before answering the last question, let us consider a couple of examples.

Ex ①. $f(z) = \sin \pi z$. This is an entire function w/ zeros at $z = n \in \mathbb{Z}$. If you single out the zero at $z=0$ and try to express

$$\sin \pi z = z \prod_{n \neq 0} (z-n)\, g(z),$$

you realize that, at e.g. $z=0$, the

infinite product $\prod_{n \neq 0}(n)$ is trying to multiply together all nonzero integers, which you cannot make any sense of as a complex number.

The second examples shows that even if you are able to make sense of the product, it may not give you what you want.

Ex ② Consider a sequence $\{a_n\}_{n=1}^{\infty}$ in $B(0, 1/4) = G$ converging to $z = 1/4$. Suppose $\exists f \in H(G)$ s.t. f has zeros at $\{a_n\}$ (are there such? Yes, by results we will prove here.) Can we make sense of $\prod_{n=1}^{\infty}(z-a_n)$?

Well, for $z \in G$, $|z-a_n| \leq \frac{1}{2} \Rightarrow$ a partial product $\left|\prod_{j=1}^{n}(z-a_j)\right| \leq \frac{1}{2^n} \to 0$ as $n \to \infty$, so setting $f_n(z) = \prod_{j=1}^{n} (z-a_n)$ gives a sequence $f_n \in H(G)$ w/ zeros at $z = a_1, \dots, a_n$, but f_n converges in $H(G)$ to 0, not to f.

<u>Lecture 17 (2/16/22).</u>

With this in mind, we first study convergence of infinite products of complex numbers in a suitable sense.

<u>Def.</u> ① A seq. $\{z_n\}_{n=1}^{\infty}$ of complex numbers has an <u>infinite product</u>,

$$z = \prod_{n=1}^{\infty} z_n,$$

if $z = \lim_{n\to\infty} \prod_{j=1}^{n} z_j$.

In view of Ex ②, we want a criterion s.t. if no $z_n = 0$, then $z \neq 0$.

Obs. • If $\Pi_n = \prod_{j=1}^{n} z_j$, $\Pi_n \to z \neq 0$,

then $z_n = \frac{\Pi_n}{\Pi_{n-1}} \to 1$. Thus, there

is no loss of generality to assume $z_n \to 1$.

We may and we shall assume $\operatorname{Re} z_n > 0$.

Note that $\operatorname{Log} z_n$ is then well defined.

principal branch.

• $\Pi_n = z_1 \cdots z_n = \exp(\operatorname{Log} z_1 + \cdots + \operatorname{Log} z_n)$

Prop 1. Let $\operatorname{Re} z_n > 0$. Then $\prod_{j=1}^{\infty} z_j$

converges to $z \neq 0$ $\Leftrightarrow$

$\sum_{j=1}^{n} \operatorname{Log} z_j$ converges to l, $z = e^{l}$.

Pf. "$\Leftarrow$" is easy and left to DIY.

"$\Rightarrow$". Write $z = e^{l}$, where

$l = \log|z| + i\theta$, $\theta \in [-\pi, \pi)$, and similarly $\Pi_n = e^{l_n}$ w/ $l_n = \log|\Pi_n| + i\,\theta_n$, $\theta_n \in [\theta-\pi, \theta+\pi)$.

Moreover, let $s_n = \sum_{j=1}^{n} \operatorname{Log} z_n$. Then,

$$\Pi_n = e^{s_n} \Rightarrow s_n = l_n + 2\pi i k_n, \; k_n \in \mathbb{Z}.$$

The l_n are chosen s.t. $\Pi_n \to z \Rightarrow l_n \to l$. In particular, by obs. above,

$$\operatorname{Log} z_n = s_n - s_{n-1} \to 0 \quad (z_n \to 1).$$

Thus, $l_n - l_{n-1} + 2\pi i (k_n - k_{n-1}) \to 0$

Since $l_n \to l$, $l_n - l_{n-1} \to 0 \Rightarrow$ $k_n - k_{n-1} \to 0$ but since $k_n \in \mathbb{Z}$, this can only happen if $k_n = k_{n-1}$ for $n >> 1$. Thus, $s_n = l_n$ for $n >> 1$ and $l_n \to l \Rightarrow s_n \to l$ as desired.

$\boxtimes$

Prop 2. Suppose $z_n = 1 + w_n$, $w_n \to 0$. Then $\sum_{n=1}^{\infty} \log z_n$ conv. abs. $\Leftrightarrow$ $\sum_{n=1}^{\infty} w_n$ conv. absolutely.

Pf. Since $f(w) = \log(1+w)$ is analytic in $\operatorname{Re} w > -1$, $f(0) = 0$, $f'(0) = 1$, we have

$$f(w) = w\,g(w), \quad g(0) = 1,$$

g analytic in $\operatorname{Re} w > -1$. It follows that $\exists\, B(0,\delta)$ s.t. $\frac{1}{2} \le |g(w)| \le \frac{3}{2}$ in $B(0,\delta) \Rightarrow$

$$(1) \qquad \tfrac{1}{2}|w| \le |\log(1+w)| \le \tfrac{3}{2}|w|, \quad w \in B(0,\delta).$$

Since $w_n \to 0$, $w_n \in B(0,\delta)$ for $n \gg 1$ and Prop 2 follows easily. ∎

Def. ②. If $z_n \to 1$, then we say that $\prod_{n=1}^{\infty} z_n$ converges absolutely if $\sum_{n=1}^{\infty} (1 - z_n)$ conv. abs. ($\Rightarrow \prod_{n=1}^{\infty} z_n = e^s$, where $s = \sum_{n=1}^{\infty} \log z_n$.)

<u>Thm 1.</u> Let $G \subseteq \mathbb{C}$ be region, $\{f_n\}$ seq. in $H(G)$ s.t. no $f_n \equiv 0$, and assume that $\sum_{n=1}^{\infty}(1-f_n)$ conv. abs. + unif. (normally) on compact subsets of G. Then ① $\prod_{n=1}^{\infty} f_n$ converges to f in $H(G)$ and ② any zero of f is a zero of finitely many of the f_n. The multipl. of any zero is the sum of the multipl. of the f_n's that are zero at the point.

<u>Pf.</u> ① Let $\underbrace{}_{}$ ($\bar{U} \subseteq G$ open s.t. $K = \bar{U} \subset\subset G$). Since $\sum_{n=1}^{\infty}(1-f_n)$ conv. abs. + unif., $f_n \to 1$ unif. on $K \Rightarrow \exists N$ s.t. $\operatorname{Re} f_n > 0$ on K ($\Rightarrow f_n \neq 0$)

Let $f_K = \prod_{n=1}^{N} f_n$. Now, by Prop 2 (estimate (1)), $\sum_{n=N+1}^{\infty} \operatorname{Log} f_n$ conv. abs. + unif. on K.

$\Rightarrow \exists\, g_K \in H(U) \cap C(K)$ s.t.

$g_K = \sum_{N+1}^{\infty} \mathrm{Log} f_n$. By cont. of $w \to e^w$,

for $z \in K$, $\prod_{j=1}^{n} f_j(z) = f_K(z)$.

$$\prod_{j=N+1}^{n} f_j(z) = f_K(z) \exp \sum_{j=N+1}^{n} \mathrm{Log} f_j(z)$$

$$\to f_K(z)\, e^{g_K(z)} = f(z).$$

Note: While f_K, g_K depend on K, $f(z)$ does not.

What about unif. convergence of $\prod_{j=N+1}^{n} f_n$? Let $\Pi_n = \prod_{j=N+1}^{n} f_j$. Then $(m \le n)$:

$$|\Pi_n - \Pi_m| = \prod_{j=N+1}^{m} |f_j| \left| \prod_{j=m}^{n} f_n - 1 \right| \quad (2)$$

Since $\sum \log f_n$ conv. abs + unif. on $K \subset\subset G$, $\exists\, M > 0$ s.t. $\sum_{n=N+1}^{\infty} \log |f_n| \le M$ on K (*)

$(g_K = ?)$

For $\varepsilon > 0$ $\exists\, \delta > 0$ s.t. $|e^w - 1| \le e^{-M} \varepsilon$ when $|w| < \delta$.

(*) Note that the bound uses also compactness of K.

Again by abs + unif. conv. of $\sum_{N+1}^{\infty} \log f_j$ we can find $N' \geq N$ s.t.

$$\sum_{j=N'+1}^{\infty} \log|f_j| < \delta \quad \text{on } K.$$

$\Rightarrow$ by (2):

$$|\Pi_n - \Pi_m| \leq e^{\sum_{N+1}^{m} \log|f_j|} \left| e^{\sum_{m}^{n} \log|f_j|} - 1 \right|$$

$$< e^{M} \cdot e^{-M} \varepsilon = \varepsilon.$$

This proves $\Pi_n \to f$ unif. on K.
Since $U \subseteq G$ was arbitrary, $\Pi_n \to f$ in G.

(2) For $U \subseteq G$ as above, $K = \overline{U} \subset\subset G$,

$$f(z) = \left(\prod_{j=1}^{N} f_j(z) \right) e^{g_K(z)}, \quad \text{and}$$

hence all zeros of f in U come from the finite product $\prod_{j=1}^{N} f_j(z)$.

The conclusion follows. $\boxtimes$